I have been asked to comment on the current attempt by Alabama to create high school mathematics standards. About one year ago, I was asked by Ted Rebarber to review a draft of this current document , and, in particular, the course entitled "'Geometry and Statistics" contained in it. What I said was "here is my very strong recommendation. Abandon this approach. Find a rock solid mathematician who is willing to help you write reasonable standards." I did not want to come out and say what I really felt which was that the so called authors of this document are functionally illiterate in mathematics, having no real idea of the way all the different areas within math interconnect, what the key topics within each are, and how much time and effert has to go into teaching them.

I've now gone through the file that Ted Rebarber sent me containing what I understand to be the final version of these standards. I paid particular attention to the new first course "Geometry with Data Analysis."

Unfortunately, the standards for this course were virtually identical to those for the Geometry and Statistics course that I criticized so strongly before. The form of the document that I received consisted of boxes, each with a sub-box on the left containing a description of what I interpreted to be the objections of the standards in the right hand sub-box. Each of these objectives is identical to a corresponding description for a sequence of standards that was in the original document I reviewed at the beginning of this year. So the objectives were not changed except that I found perhaps two of the orgiginal objectives had been deleted - though it seemed their standards had simply been moved to other boxes.

The standards were, with only a few exceptions, identical to those in the previous document, and all the objections I had before applied equally well to the "new" course.

Moreover, in a fit of masochism, I actually looked more closely at a number of these "standards." I particularly "liked" the standard in the second box on page 112: 8. Use technology to organize data, including very large data sets, into a useful and manageable structure. First, what does this have to do with geometry? Second, what will students learn about actually organizing data so as to learn what it might be able to reveal to them when they "use technology." All too often, the very use of general or even specialized programs to sort data completely hides what the students need to see. Before this, students need to really study many examples of how key information hides in data. Moreover, to do this for real - and it is one of the most crucial things students have to learn in their math courses - takes a huge amount of time and effort. This one sentence standard. hidden and horribly distorted in this horrid course, should be the heart of any real course in data analysis.

And then, for the first box on page 113 we have "13. Compute (using technology) and interpret the correlation coefficient of a linear relationship."

Correlation coefficient? How is this going to be handled? Are the students expected to know what this number actually is, and how and why it is defined in the way that it is? The background required for any of this is pretty darned massive: lots of advanced linear algebra, quadratic forms, and even measure theory, which is advanced college level material.

And how about the first box on page 114? Here there are two standards:

19. Derive and apply the relationships between the lengths, perimeters, areas, and volumes of similar figures in relation to their scale factor.

20. Derive and apply the formula for the length of an arc and the formula for the length of a sector.

Let me discuss the second item. The first is even more inappropriate. What the authors expect is that the students will know that the circumference of a circle is $2\pi r$, and an arc subtending an angle θ from the center will, "of necessity" cover $\frac{\theta}{2\pi}$ of the circumference, so the answer (in radians) is θr. However, we should keep in mind that the teacher should be able to answer the kinds of questions that smarter students will be likely to ask, and chief among them is "how do we measure the length of a curve that is not an arc of a circle or a segment of a straight line?"

In fact, it is the existence of this question which is the main reason our "classical" geometry courses didn't talk about arc length at all or, if they did, they only did it at the very end of the year. Here is what the teacher needs to know to even begin to answer that question: first, one needs to know how to describe a reasonably general curve in the plane - they have to be able to introduce the parametric equation for a curve, $((x, y) = (f(t), g(t))$ - and then, at least heuristically, what the integral $\int_a^b \sqrt{\frac{df}{dx}^2 + \frac{dg}{dt}^2}dt$ means and how to evaluate it.

Moreover, all this does not even begin to handle the use of the terms **derive and apply**. How could the authors have so little understanding of the subject so as to give this requirement without strictly pointing out how to limit it to a level that at least has a chance of being possible for ninth grade students?

The remaining standards for this course are basically unchanged from the Draft standards that I originally evaluated for you. In my discussion I took pains to give you a detailed explanation of the history of the approach to geometry that these standards represent, and the fact that it has never been successful for any but the most talented and determined high school students to handle even in the countries with the best mathematics outcomes in the world. And you need to understand that we are nowhere near that level.

Some comments on the glossary.

One of the fastest ways of determining the depth of the mathematical knowledge of the authors of mathematics standards like these is the handling of the terms in the included glossary. In the present case, the results are again horrifying. Here are some examples:

- **Pythagorean Theorem:** The sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse. *Comments: The legs of a right triangle are* **line segments***. They are not numbers, and I don't really know what it might mean to take the square of a line segment and getting a number as the result. What they should have said was that the sum of the squares of the* **lengths** *of the two legs is equal to the square of the* **length** *of the hypotenuse.*

- **Ratio:** The multiplicative comparison of two non-zero quantities. Represented as $a{:}b$, a/b, and a to b. *Comments: incoherent. Also incorrect. First, quantities should be relaced by numbers. But we should also note that "number" is not defined in the glossary. This is actually not surprising since, a number of years ago, in a national meeting with math educators I was asked to run a seminar on basic mathematics, so I started out by asking the audience to tell me what a number is. I was greeted with dead silence, and after trying to get any kind of meaningful remarks from them, I finally had to tell them that numbers are objects that have two binary operations, the first written* $+$ *and the second* $\times$ *with the usual properties of commutativity, associativity, additive inverses, and multiplicative cancellation,* $a \times b = a \times c$ *with* $a \neq 0$ *implies* $b = c$*.*

- **Irrational number:** A number that cannot be written as a ratio of two numbers, including non-terminating and non-repeating decimals (such as π). *Comments: EVERY number can be written as a* **quotient** *of two numbers, provided that your number set has an element* 1 *(with the property that* $1 \times n = n$ *for all* n *in the number set). Just take* $n/1$*. So we see that the definition given cannot be correct. The correct definition is, a* **Real** *number* r *is irrational if and only if there are no integers* a *and* b *with* $b \neq 0$*, so that* $r \times a = b$*.*

I could continue in this way for a really huge number of the items in this indescribably bad glossary, but I hope the point has been made that this document doesn't merely need corrections. It needs to be completly redone essentially from scratch, but by a group that actually knows mathematics.

The remainder of this note is the review that I sent you then, and the recommendation that I gave you then remains even more my view for the current document.

I have been asked to comment on the very strange high school math curriculum that starts with a course on Geometry and Statistics. In summary, here is my very strong recommendation. Abandon this approach. Find a rock solid mathematician who is willing to help you write reasonable standards that are actually teachable in K-12 such as Prof. Larry Gray, Emeritus Professor of Mathematics, University of Minnesota, who is largely responsible for the very solid Minnesota K-12 mathematics standards, and follow his advice.

What follows are my detailed comments:

The introduction starts as follows:

> Geometry with Statistics is a new course developed for inclusion in the 2019 Alabama Course of Study:
>
> "Mathematics. It is the first course in high school mathematics, taken by all students in Grade 9, giving them access to the same mathematics and building on their experiences in the middle grades.
>
> "Geometry with Statistics is important for the development of mathematical knowledge and skills through visual representations prior to the more abstract development of algebra. Leading with Geometry with Statistics in Grade 9 offers high school students the opportunity to build their reasoning and sense-making skills, see the applicability of mathematics, and better prepare for further studies in algebra.

It is astounding that anyone with any real understanding of mathematics could possibly think to ever combine these two topics into a single course. They have literally nothing in common, especially at the K-12 level.

For background, let me give you the real reasons that, for a large number of years, there has even been pressure on the states to include the topics of data analysis, statistics, and probability in the K-12 math curriculum. Since the mid 1940s people in the federal government have pushed for more broad-based understanding of this material, and there were real reasons for doing so. However, the real reasons for this were classified and had to do with certain things that happened during the second world war.

The reasons were explained to me more than 30 years ago by two of my now deceased colleagues at Stanford, Professors Ralph Philips and David Gilbarg. They had been assigned to the arm of the Navy that was involved in the war on the Pacific Ocean, and were asked to help with what they told me was the main problem in that theater: finding the location of the Japanese fleet every day after they had moved during the cover of night. In hindsight, looking at Philips and Gilbargs mathematical publications, one can see both were regarded as among the top experts in the world in the areas of probability and statistics, in particular, the application of techniques in partial differential equations to these areas.

In any case, they were very successful in applying their techniques to the original problem, and it was felt that our superiority in this area was crucial to our winning the war. So it was natural that the national government would try to get more citizens with expertise in these areas. The mathematics seemed so incredibly powerful, but those government people knew very little about the actual subject, as was clearly the case with whomsoever wrote the Alabama document that we are looking at here. As a result, they seemed to feel that the best way to do this was to introduce these topics into K-12.

Of course, as any knowledgeable mathematician would be able to explain to you, the material that was used for this dramatic success was research level mathematics at that time, and it remains so today. In fact, to even begin to do these kinds of things in any meaningful way, students have to be totally fluent with the most advanced areas of linear algebra, advanced calculus, and partial differential equations. So the fact is that all this push nationally to develop expertise in statistics and probability has had virtually no real effect. Rather, it seems to have created a sub-population that claims expertise in the area and, in actual fact, has absolutely no idea of what is actually going on.

(Parenthetically, I and the son of one of the mathematicians involved in the work I described above, Alan Tucker, were on the commission appointed by Achieve to rough in the structure of the Common Core mathematics standards, and both of us tried quite hard to give the other members of the commission a more realistic understanding of what kinds of mathematics should precede attempting to have students develop the ability to handle sensible material in statistics and probability. Unfortunately, we were not successful.)

So, in the Alabama document, the treatment of statistics in the horrible Geometry and Statistics course is low level, has absolutely no connection with the geometry, and essentially guarantees the continuing total lack of success of Alabamas K-12 mathematics courses in areas that are not only connected to statistics and probability, but most likely, also geometry.

Now let us turn to the mathematics in this course. Well before any mention of geometry, we have **the main objectives of the statistics material:**

Mathematical and statistical reasoning about data can be used to evaluate conclusions and assess risks.

- *Use mathematical and statistical reasoning with quantitative univariate data (set of values) or bivariate data (set of pairs of values) that suggests a linear association, in order to evaluate conclusions and assess risk. For example, estimate the typical age at which a cancer patient is diagnosed, and estimate the how the typical age differs depending on the number of cigarettes smoked per day.*

Making and defending informed data-based decisions is a characteristic of a quantitatively literate person.

- *Make and defend a decision based on an analysis of univariate and bivariate quantitative data that suggests a linear association.*

Examples: 1. Refer to shape, center, and spread of ticket prices for comparable events to set and justify the price to an event.

2. Estimate a line of fit to predict the population of a country based on a set of historical data.

Focus 2: Visualizing and Summarizing Data

Data arise from a context and come in two types: quantitative (continuous or discrete) and categorical. Technology can be used to clean and organize data, including very large data sets, into a useful and manageable structurea first step in any analysis of data.

- *Distinguish between quantitative (continuous or discrete) and categorical data and between the techniques used for analyzing data of these two types. Distributions of quantitative data (continuous or discrete) in one variable should be described in the context of the data with respect to what is typical (the shape, with appropriate measures of center and variability, including standard deviation) and what is not (outliers), and these characteristics can be used to compare two or more subgroups with respect to a variable.*

- *Represent data with plots on the real number line, choosing a format (dot plots, histograms, and box plots) most appropriate to the data set; do simple cases by hand and more complex cases using technology. [S.ID.1 edited]*

The words used here are fancy, bivariate data, distributions, curve fitting, etc., but, for example, the actual meaning of distribution in statistics and probability was one of the main problems in the area from the late 19th century to the middle of the 20th, and the proper definition requires a sophisticated understanding of Lebesgue integration and measure theory. When Alan and I tried to explain this to other members of the Achieve commission mentioned above, even though the group included one other supposedly competent mathematician, we consistently got the thousand yard blank, panic stare.

These standards then continue in exactly the same manner:

- *Use statistics appropriate to the shape of the data distribution to compare and contrast two or more data sets by considering center (median, mean) and spread (interquartile range). [S.ID.2]*

- *Interpret differences in shape, center, and spread in the context of the data sets,* *accounting for possible effects of extreme data points (outliers). [S.ID.3]*

Scatterplots, including plots over time, can reveal patterns, trends, clusters, and gaps that are useful in analyzing the association between two contextual variables.

- *Represent data on two quantitative variables using scatterplots, including plots over time, in order to explore patterns, trends, clusters, and gaps useful in analyzing the association between two contextual variables.*

Analyzing the association between two quantitative variables should involve statistical procedures, such as examining (with technology) the sum of squared deviations in fitting a linear model, analyzing residuals for patterns, generatingleast-squares regression line and finding a correlation coefficient, and differentiating between correlation and causation.

- *Fit a linear function for a scatter plot that suggests a linear association. Informally assess the fit of the lineplotting and analyzing residuals. [S.ID.6.c]*

Data-analysis techniques can be used to develop models of contextual situations and to generate and evaluate possible solutions to real problems involving those contexts.

- *Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data. [S.ID.C.7]*

- *Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each. [S.IC.3]*

Again, the concepts being casually thrown about such as shape of the data distribution, sum of squares deviation, least square regression, even the idea of fitting to a model, (how do you measure closeness?) are things that are very, very difficult to define and handle in a sensible way. The thing is that, for example, over 100 years ago a very clever man, J. L. F. Bertrand, studied examples of such problems and showed that one got entirely distinct results depending on how you measure closeness. His work led directly to the necessity for requiring that serious students in these areas must understand advanced linear algebra, and very advanced material in the theories of measure and integration that, even today are key areas all graduate students must understand who wish to work in these areas. Alan and I felt that it might well be possible to structure things so that students could actually work through maybe one or two of the Bertrand counterexamples by the time they graduated high school, and this would be a tremendous help to them later, but to essentially randomly learn sequences of words and what buttons to push on a scientific calculator to get numbers somehow associated to those words, is simply not going to be helpful out in the real world.

Now, let us turn to the main topics covered in the geometry part of this material.

Areas and volumes of figures can be computed by determining how the figure might be obtained from simpler figures by dissection and recombination.

- Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects. [G.GMD.4]

- Give an informal argument for the formulas for the circumference of a circle, area of a circle, and volume of a cylinder, pyramid, and cone. Use dissection arguments, Cavalieris principle, and informal limit arguments. [G.GMD.1]

- Model and solve problems using surface area and volume of prisms, cylinders, pyramids, cones, and spheres including composite solids and solids with portions removed. (edited) [G.GMD.3]

Note, once more the fancy words dissection and recombination. What the authors really mean is that one can SOMETIMES cut up figures in the plane into unions of triangles with disjoint interiors, and when you can do this you can determine the area by adding up the areas of the triangles. (Of course, this can only happen of the boundary of the figure consists entirely of line segments.) But this is material that actually is typically shown to students in much earlier grades.

Then in the very first bullet standard, the authors take this and try to make it relevant to calculating volumes, but there are a huge number of totally distinct types of figures that can occur by rotating even the rectilinear figures discussed above about lines. Suppose, for example, the line DOES NOT LIE IN the plane of the figure. But even if it does lie in the plane of the figure, there are still huge numbers of possibilities depending on whether and how the line intersects the figure. All this needs to be carefully spelled out, and the exact level at which the material is introduced and covered has to be explained.

The types of issues I mentioned in the last paragraph are issues that very good mechanical engineers have to spend quite a bit of time understanding, and things get very advanced very quickly. So this sketchy material is absolutely unacceptable as it stands.

Of course, the authors blindly continue with this kind of thing:

Constructing approximations of measurements with different tools, including technology, can support an understanding of measurement.

- Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line. [G.CO.12]

- Make and test conjectures about what formal geometric constructions are needed to produce a given shape.

Example: Identify and describe two different ways to construct a rhombus.

- Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle using technology and other tools. [G.CO.3. edited]

- Use coordinates to compute perimeters and areas of polygons using a variety of methods, e.g., using the distance formula. [G.GPE.7]

When an object is the image of a known object under a similarity transformation, a length, area, or volume on the image can be computed by using proportional relationships.

- Derive and apply the proportional relationship between similar figures and their lengths, areas, and volumes.

• Derive and apply the formula for the length of the arc and the formula for the area of a sector. [G.C.5 edited]

Look at the Example Identify and describe two different ways to construct a rhombus. What does different mean here? Then, look at the next standard construct an equilateral triangle ... The key words here are using technology. In other words the student will not be required to have any real idea about WHY the constructed figures are really equilateral triangles, etc. The answer will be 'because my scientific calculator said so!'

Now look at the last two standards above. Generally, the meaning of proportional relationship involves LINEAR concepts. But areas vary as to squares, and volumes vary as to cubes. And, finally, the last standard above. What could possibly be meant by derive here? What is the student to assume, how are the assumptions to be justified? Since students are not dealing with linear situations what kinds of things are they supposed to understand. In real life, when dealing with things like this we introduce symmetry and are usually forced to talk about GROUPS. In fact, what engineers have to understand is considerable information about the groups of ORTHOGONAL matrices (either 2 by 2 or 3 by 3). But this is a huge deal.

Unfortunately, the authors continue on and next introduce orthogonal transformations and rigid translations (but rigid translations actually require 4 by 4 matrices to be handled precisely). But the introduction is extremely imprecise And things continue in this vein.

Applying geometric transformations to figures provides opportunities for describing the attributes of the figures preserved by the transformation and for describing symmetries by examining when a figure can be mapped onto itself.

- Represent transformations in the plane (coordinate and otherwise) using, e.g., tracing paper and geometry software. [G.CO.2]

a. Describe transformations as functions that take points in the plane as inputs and give other points as outputs using informal and formal notation.

b. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

- Given a polygon, describe the rotations and/or reflections that carry it onto itself, if possible. [G.CO.3]

  a. Describe the attributes of a figure that can be mapped onto itself in a single transformation.

- Develop definitions of rotation, reflection, and translation in terms of angles, circles, perpendicular lines, parallel lines, and linesegments. [G.CO.4 edited]

Note the totally heuristic definitions (tracing paper, geometry software).

It should be clear that just the geometry material above, to be handled reasonably well, should take well over a year. But the approach basically through continuous transformation groups is extremely problematic in K-12. There is virtually no evidence that it works when taken to scale. Indeed, fifty years ago, in the old USSR, at the peak of their development of the K-7 mathematics courses that today are the foundation of the courses in these grades in ALL the high achieving countries, (the United States is conspicuously absent from this list), the math educators in the USSR introduced a new high school geometry course that almost exactly followed the lines of the standards above.

It was a disaster, and was rapidly replaced with their old geometry courses. And here, in this country, the approach being used in the Alabama document has been even more of a disaster.

On the other hand, if I were to teach a geometry course to advanced undergraduate math majors at Stanford, then a corrected approach, but associated to the approach in the Alabama document, is exactly how I would proceed: first, introduce the key transformation groups together with their realization as matrix groups, and then develop the associated geometries as representations of the groups. But there is no way that I would even think of handling things this way with more ordinary students even at the college level.

However, the authors of the Alabama document, not content with the above material already far beyond the capabilities of most students - go even further, introducing even more material that should be handled in another at least year long course.

ARE YOU SURE YOU REALLY WANT TO DO THIS? Let me reiterate my recommendation in the strongest terms. Put your current document in your circular file.